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Canonical orthonormal basis for SU(3) ⊃ SO(3). II: Reduced matrix elements of the SU(3) generators

R Le Blanc and D J Rowe

Department of Physics, University of Toronto, Toronto, Ontario, Canada, M5S 1A7

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Abstract. A simple algorithm is given for the calculation of reduced matrix elements of the generators of the su(3) algebra in the canonical SU(3) ⊃ SO(3) basis introduced in I.

1. Introduction

An unambiguous and group theoretically sound labelling scheme has recently been proposed by Deenen and Quesne (1983, see also Quesne 1984a, b) to solve the labelling problem for the SU(n) ⊃ SO(n) group chain. This scheme was successfully used in part I of this series of papers (Le Blanc and Rowe 1985) to build canonical orthonormal bases for SU(3) ⊃ SO(3) representations. Since the group chain SU(3) ⊃ SO(3) is of widespread importance in, for instance, the nuclear problem (Elliott 1958a, b, Arima and Iachello 1976) and the three body problem (Chacón *et al* 1984), the calculation of reduced matrix elements of the generators of the su(3) algebra and, more generally, of SU(3) tensor operators is of major interest.

The purpose of this paper is to present a simple algorithm for the calculation of reduced matrix elements of the generators of SU(3) in the new canonical basis. Since the actual construction of the basis uses the complementarity of the O(3) and the Sp(2, $\mathfrak{R}$) group actions within the fundamental representations of the Sp(6, $\mathfrak{R}$) group (Le Blanc and Rowe 1985, Deenen and Quesne 1983), we proceed by calculating the matrix elements of the fundamental Sp(2, $\mathfrak{R}$) × O(3) tensor from which all other matrix elements can be constructed by a simple building up process.

Finally note that the techniques introduced in this paper can be generalised to the problem of mixing Sp(3, $\mathfrak{R}$) × O(A) irreps; a problem of relevance to the microscopic theory of nuclear collective shell structure (Rosensteel and Rowe 1980) and it is our intention to present an investigation of such a generalisation in a subsequent publication.

2. Complementarity of the sp(2, $\mathfrak{R}$) ⊃ u(2) and the su(3) ⊃ so(3) algebras

As shown in I, canonical bases for SU(3) ⊃ SO(3) unirreps are easily constructed using the well known complementarity that exists in a six-dimensional Bargmann space between the Sp(2, $\mathfrak{R}$) and O(3) actions, on the one hand, and between the U(2) and U(3) actions, on the other.

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A realisation of the sp(2, $\Re$) algebra is given in terms of the six complex (Bargmann) variables $g_{\alpha i}$, $\alpha = 1,2$, $i = 1,3$, by the ten generators

$$\mathscr{A}_{\alpha\beta} = \boldsymbol{g}_\alpha \cdot \boldsymbol{g}_\beta \tag{2.1a}$$

$$\mathscr{B}_{\alpha\beta} = \partial^2/\partial g_{\alpha i}\, \partial g_{\beta i} \tag{2.1b}$$

$$\mathscr{C}_{\alpha\beta} = g_{\alpha i}\, \partial/\partial g_{\beta i} + \tfrac{3}{2}\delta_{\alpha\beta} \tag{2.1c}$$

with summation over repeated indices. The u(2) subalgebra is spanned by the four ($\mathscr{C}_{\alpha\beta}$) generators.

Likewise, a realisation of the u(3) algebra is given by the nine generators

$$C_{ij} = g_{\alpha i}\, \partial/\partial g_{\alpha j}. \tag{2.2}$$

A convenient basis for the eight-dimensional su(3) algebra is given by the spherical tensors

$$L_m = C_{1m} = -\sqrt{2}\,\langle 1m'; 1m''|1m\rangle g_{\alpha m'}\, \partial/\partial \bar{g}_{\alpha m''} \tag{2.3a}$$

and

$$C_{2\nu} = \sqrt{2}\,\langle 1m'; 1m''|2\nu\rangle g_{\alpha m'}\, \partial/\partial \bar{g}_{\alpha m''} \tag{2.3b}$$

where

$$g_{\alpha m} = (-1)^m \bar{g}_{\alpha -m}.$$

L is the angular momentum and $Q_2 = \sqrt{3}\, C_2$ is the su(3) quadrupole tensor.

As shown in I, basis states for an Sp(2, $\Re$) unirrep $\langle\sigma\rangle = \langle\sigma_1\sigma_2\rangle$ can be indexed by the subgroup chain labels

$$\begin{array}{cccc} \mathrm{Sp}(2,\Re) & \supset \mathrm{U}(2) & \supset U(1) \\ \langle\sigma\rangle & (n)\,\{\omega\} & \nu \end{array} \tag{2.4}$$

where $(n) = (n_1, n_2)$ are missing labels and $\{\omega\} = \{\omega_1, \omega_2\}$. On the other hand, basis states for a U(3) unirrep $\{h\} = \{h_1, h_2, h_3\}$ can be labelled by

$$\begin{array}{cccc} \mathrm{U}(3) & \supset \mathrm{O}(3) & \supset \mathrm{SO}(3) & \supset \mathrm{SO}(2) \\ \{h\} & (n')[L\varepsilon] & L & M \end{array} \tag{2.5}$$

where $n' = (n_1', n_2')$ again are missing labels, $\varepsilon = 0$ or 1 and $L \geqslant \varepsilon$.

Since the above Sp(2, $\Re$) and U(3) actions commute, it follows that basis states for the six-dimensional Bargmann space can be simultaneously indexed by the labels of both subgroup chains. However only a subset of possible representations occur in this space. In particular, as a consequence of the Sp(2, $\Re$) $\times$ O(3) complementarity (Moshinsky and Quesne 1970, 1971), only representations with

$$\langle\sigma\rangle = \langle L + \tfrac{3}{2}, \varepsilon + \tfrac{3}{2}\rangle \equiv \langle\tfrac{3}{2}(L\varepsilon)\rangle \tag{2.6}$$

occur. And as a consequence of the U(2) $\times$ U(3) complementarity (Biedenharn and Giovanni 1967), only the two-rowed U(3) representations (i.e., $h_3 = 0$) occur with

$$\{\omega\} = \{h_1 + \tfrac{3}{2}, h_2 + \tfrac{3}{2}\} \equiv \{\tfrac{3}{2}(h)\}. \tag{2.7}$$

It is also well known that the Sp(2, $\Re$) $\times$ O(3) and U(2) $\times$ U(3) representations are multiplicity free in this space so that, as shown in I, the two sets of multiplicity indices can be identified; i.e.,

$$(n) \equiv (n'). \tag{2.8}$$

Thus the two subgroup chains share their labels and an orthonormal basis for the six-dimensional Bargmann space is given by states labelled

$$|\{h\}(n)[L\varepsilon];\ \nu M\rangle. \tag{2.9}$$

3. Construction of an orthonormal basis

Let

$$|\langle L\varepsilon\rangle;\ \nu M\rangle = |[L\varepsilon];\ \nu M\rangle \equiv |\{L\varepsilon\}(00)[L\varepsilon];\ \nu M\rangle \tag{3.1}$$

denote a member of the $Sp(2,\Re)$ lowest multiplet of $U(2)\times U(3)$ states; i.e. the states satisfying

$$\mathscr{B}_{\alpha\beta}|[L\varepsilon];\ \nu M\rangle = 0. \tag{3.2}$$

These states are given by the Bargmann wavefunctions

$$\psi^{[L]}_{\nu M} \equiv \langle g|[L];\ \nu M\rangle = [\mathscr{Y}^{L/2+\nu}(g_1)\times\mathscr{Y}^{L/2-\nu}(g_2)]^L_M, \tag{3.3}$$

$$\psi^{[L1]}_{\nu M} = \left(\frac{2}{L+1}\right)^{1/2}[\psi^{[L-1]}_{\nu}(g)\times\psi^{[11]}(g)]^L_M \tag{3.4}$$

where $\mathscr{Y}^L_M$ is proportional to a solid harmonic

$$\mathscr{Y}^L_M(\boldsymbol{r}) = \left(\frac{4\pi}{(2L+1)!!}\right)^{1/2} r^L Y^L_M(\theta,\varphi), \tag{3.5}$$

and

$$\psi^{[11]}_{0M}(g) = (\mathrm{i}/\sqrt{2})\mathscr{Y}^1_M(g_1\wedge g_2). \tag{3.6}$$

Thus, for example,

$$\psi^{[L]}_{L/2L}(g) = (1/\sqrt{L!})(g_{1+1})^L \tag{3.7}$$

$$\psi^{[11]}_{01}(g) = (1/\sqrt{2})(g_{1+1}g_{20} - g_{10}g_{2+1}) \tag{3.8}$$

where

$$g_{1\pm1} = \mp(1/\sqrt{2})(g_{11} + \mathrm{i}g_{12}). \tag{3.9}$$

A non-orthonormal basis

$$|\Psi(\{h\}(n)[L\varepsilon];\ \nu M)\rangle = [Z^{\{n\}}(\mathscr{A})|[L\varepsilon];\ M\rangle]^{\{h\}}_{\nu} \tag{3.10}$$

is now given by the U(2) coupling of a polynomial $Z^{\{n\}}(\mathscr{A})$ of U(2) rank $\{n\}$ in the symplectic raising operators with the above $Sp(2,\Re)$ lowest weight states. The polynomials $Z^{\{n\}}(\mathscr{A})$ are defined by

$$Z^{\{n\}}(\mathscr{A}) = N(x,y)\mathscr{Y}^x(\mathscr{A})(2\mathscr{A}_{+1}\mathscr{A}_{-1} - \mathscr{A}_0^2) \tag{3.11}$$

with

$$N(x,y) = \left(\frac{(2x+1)!(x+y)!}{x!y!(2x+2y+1)!}\right)^{1/2},$$

$$x = \tfrac{1}{2}(n_1 - n_2),$$

$$y = \tfrac{1}{2}n_2.$$

This polynomial is given in the Bargmann representation by the substitution

$$\mathscr{A}_{+1} = (1/\sqrt{2})g_1 \cdot g_1, \qquad \mathscr{A}_0 = g_1 \cdot g_2, \qquad \mathscr{A}_{-1} = (1/\sqrt{2})g_2 \cdot g_2. \tag{3.12}$$

Finally, the states (3.10) can be transformed into the orthonormal basis (2.9) by means of a simple Hermitian transformation

$$|\{h\}(n)[L\varepsilon];\ \nu M\rangle = K^{-1}(\{h\}[L\varepsilon])|\Psi(\{h\}(n)[L\varepsilon];\ \nu M)\rangle$$

where $K(\{h\}[L\varepsilon])$ is the Hermitian square root of the overlap matrix

$$K^2_{nn'}(\{h\}[L\varepsilon]) = \langle\Psi(\{h\}(n)[L\varepsilon])\,|\,\Psi(\{h\}(n')[L\varepsilon])\rangle. \tag{3.13}$$

This transformation is easily calculated following the appendix of I.

4. Reduced matrix elements of the fundamental tensor

Matrix elements of the sp(2, $\mathfrak{R}$) and su(3) algebras, and indeed of any polynomials in the Bargmann variables $(g_{\alpha i})$, can obviously be expressed in terms of the matrix elements of these variables themselves. Furthermore, since $\partial/\partial g_{\alpha i}$ is the Hermitian adjoint of $g_{\alpha i}$ with respect to the Bargmann measure, its matrix elements are simply related to those of $g_{\alpha i}$. Now $g_{\alpha i}$ is a component of the fundamental U(2) × O(3) tensor T of U(2) rank {10} and O(3) rank [10]. On the other hand, $\partial/\partial g_{\alpha i}$ is a component of the adjoint tensor $T^\dagger$ of U(2) rank {0, −1} and O(3) rank [10]. Together, $(g_{\alpha i})$ and $\partial/\partial g_{\alpha i}$ are components of the fundamental Sp(2, $\mathfrak{R}$) × O(3) tensor of Sp(2, $\mathfrak{R}$) rank $\langle\overline{(1,0)}\rangle$ (the bar indicating that $g_{\alpha i}$ and $\partial/\partial g_{\alpha i}$ span a finite and hence non-unitary irrep of Sp(2, $\mathfrak{R}$)) and O(3) rank [10]. It is therefore of prime importance to calculate the matrix elements of the tensor $T^{\{10\}[10]} \equiv T$.

In this section, we calculate matrix elements of T reduced with respect to both U(2) and O(3) in the basis (2.9). In the following section, we then use these matrix elements to calculate matrix elements of the su(3) algebra in the canonical SU(3) ⊃ SO(3) basis.

In a subsequent paper, it will be shown how the matrix elements of T can be used to infer Sp(2, $\mathfrak{R}$) × O(3) and more generally Sp(3, $\mathfrak{R}$) × O(A) reduced matrix elements by making use of the Wigner–Eckart theorem as extended to non-compact groups by Klimyk (1983 and references therein).

We start by considering the selection rules for the doubly reduced matrix elements

$$\langle\{h'\}(n')[L'\varepsilon']\|\,T^{\{10\}[10]}\|\{h\}(n)[L\varepsilon]\rangle$$

defined by

$$\langle\{h'\}(n')[L'\varepsilon'];\ \nu'M'|\,T^{\{10\}[10]}_{\alpha\ \ m}|\{h\}(n)[L\varepsilon];\ \nu M\rangle$$
$$= \langle\tfrac{1}{2}h\nu;\tfrac{1}{2}\alpha|\tfrac{1}{2}h'\nu'\rangle \times \langle LM;1m|L'M'\rangle\langle\{h'\}(n')[L'\varepsilon']\|\,T^{\{10\}[10]}\|\{h\}(n)[L\varepsilon]\rangle, \tag{4.1}$$

where $\frac{1}{2}h = \frac{1}{2}(h_1 - h_2)$, etc.

Since $T^{\{10\}[10]}$ is of U(2) rank {10} and of angular momentum $L = 1$, we must have

$$\{h'_1h'_2\} = \{h_1+1, h_2\} \text{ or } \{h_1, h_2+1\},$$
$$L' = L, L\pm 1. \tag{4.2}$$

The selection rule for ε is obtained by recalling (cf I) that basis states only occur with

$$\begin{aligned} \varepsilon &= 0 \qquad \text{for } h_1+h_2-L \text{ even},\\ \varepsilon &= 1 \qquad \text{for } h_1+h_2-L \text{ odd}. \end{aligned} \tag{4.3}$$

This implies the selection rules

$$\begin{aligned} &L'=L+1, && \varepsilon'=\varepsilon, && n_1'+n_2'=n_1+n_2,\\ &L'=L, && \varepsilon'=\varepsilon+1=1, && n_1'+n_2'=n_1+n_2,\\ &L'=L, && \varepsilon'=\varepsilon-1=0, && n_1'+n_2'=n_1+n_2+2,\\ &L'=L-1, && \varepsilon'=\varepsilon, && n_1'+n_2'=n_1+n_2+2. \end{aligned} \tag{4.4}$$

The non-orthonormal basis states (3.10) are given in the Bargmann representation by

$$[Z^{\{n\}}(g)\psi_M^{\langle L\varepsilon\rangle}(g)]_\nu^{\{h\}} \tag{4.5}$$

where

$$Z^{\{n\}}(g)=Z^{\{n\}}(\mathscr{A}(g)).$$

Multiplying such a basis state by the fundamental tensor $T^{\{10\}[10]}(g)=g$, we can couple to states

$$[g[Z^{\{n\}}(g)\psi^{\langle L\varepsilon\rangle}(g)]^{\{h\}}]^{\{h'\}[L'\varepsilon']} \tag{4.6}$$

according to the above selection rules.

Since g and $Z^{\{n\}}(g)$ commute and since the U(2) and the O(3) couplings commute, we can effect a U(2) recoupling

$$\begin{aligned}[g[Z^{\{n\}}(g)\psi^{\langle L\varepsilon\rangle}(g)]^{\{h\}}]^{\{h'\}[L'\varepsilon']} &= \sum_k (-1)^{(\sigma+h'-h-k)/2}\mathrm{U}\left(\frac{1}{2}\frac{\sigma}{2}\frac{h'}{2}\frac{n}{2};\frac{k}{2}\frac{h}{2}\right)\\ &\quad\times[Z^{\{n\}}(g)[g\psi^{\langle L\varepsilon\rangle}(g)]^{\{k\}[L'\varepsilon']}]^{\{h'\}} \end{aligned} \tag{4.7}$$

where

$$\sigma=L-\varepsilon,$$

$$\{k\}=\{k_1,k_2\}, \qquad k=k_1-k_2, \qquad \text{etc.}$$

It follows that the important quantities to calculate are

$$[g\psi^{\langle L\varepsilon\rangle}(g)]^{\{k\}[L'\varepsilon']}. \tag{4.8}$$

Following lengthy but straightforward calculations, we find the following complete set of non-zero couplings:

$$[g\psi^{\langle L,0\rangle}(g)]^{\{L+1,0\}[L+1,0]}=[L+1]^{1/2}\psi^{\langle L+1,0\rangle}(g), \qquad L\geqslant 0, \tag{4.A1}$$

$$[g\psi^{\langle L,1\rangle}(g)]^{\{L+1,1\}[L+1,1]}=\left(\frac{L(L+2)}{L+1}\right)^{1/2}\psi^{\langle L+1,1\rangle}(g), \qquad L\geqslant 1, \tag{4.A2}$$

$$[g\psi^{\langle L,0\rangle}(g)]^{\{L,1\}[L,1]}=\left(\frac{(L+1)}{L}\right)^{1/2}\psi^{\langle L,1\rangle}(g), \qquad L\geqslant 1, \tag{4.B1}$$

$$[g\psi^{\langle L,1\rangle}(g)]^{\{L+1,1\}[L,0]}=\frac{1}{(L+1)}\left(\frac{L+2}{L}\right)^{1/2}[Z^{\{2,0\}}(g)\psi^{\langle L,0\rangle}(g)]^{\{L+1,1\}}, \qquad L\geqslant 1, \tag{4.B2}$$

$$[g\psi^{\langle L,1\rangle}(g)]^{\{L,2\}[L,0]} = \left(\frac{2(L+1)}{L}\right)^{1/2} [Z^{\{2,0\}}(g)\psi^{\langle L,0\rangle}(g)]^{\{L,2\}}, \qquad L \geqslant 2, \tag{4.B3}$$

$$[g\psi^{\langle L,0\rangle}(g)]^{\{L+1,0\}[L-1,0]} = -\left(\frac{2L}{(2L+1)(2L-1)}\right)^{1/2} [Z^{\{2,0\}}(g)\psi^{\langle L-1,0\rangle}(g)]^{\{L+1,0\}}, \qquad L \geqslant 1, \tag{4.C1}$$

$$[g\psi^{\langle L,0\rangle}(g)]^{\{L,1\}[L-1,0]} = -\frac{1}{L}\left(\frac{(2L+1)(L-1)}{(2L-1)}\right)^{1/2} [Z^{\{2,0\}}(g)\psi^{\langle L-1,0\rangle}(g)]^{\{L,1\}}, \qquad L \geqslant 2, \tag{4.C2}$$

$$[g\psi^{\langle L,1\rangle}(g)]^{\{L+1,1\}[L-1,1]} = -\left(\frac{2(L-1)(L+1)}{L(2L-1)(2L+1)}\right)^{1/2} [Z^{\{2,0\}}(g)\psi^{\langle L-1,1\rangle}(g)]^{\{L+1,1\}}, \qquad L \geqslant 2, \tag{4.C3}$$

$$[g\psi^{\langle L,1\rangle}(g)]^{\{L,2\}[L-1,1]} = -\left(\frac{(L-2)(2L+1)}{L(2L-1)(L+1)}\right)^{1/2} [Z^{\{2,0\}}(g)\psi^{\langle L-1,1\rangle}(g)]^{\{L,2\}}, \qquad L \geqslant 2. \tag{4.C4}$$

We are now in a position to calculate the reduced matrix elements of the fundamental tensor $T^{\{10\}[10]}$ in the orthonormal basis (2.9).

Note that reduced matrix elements in the orthonormal basis (2.9) can be expressed

$$\langle\{h'\}(n')[L'\varepsilon']\| T^{\{10\}[10]} \|\{h\}(n)[L\varepsilon]\rangle = (-1)^{(h+1-h')/2} \times (-1)^{L+1-L'} \langle\{h'\}(n')[L'\varepsilon'];\ \nu M|[T|\{h\}(n)[L\varepsilon]\rangle]^{\{h'\}[L'\varepsilon']}_{\nu\ M}. \tag{4.9}$$

Note too that if, in general, $\{|j)\}$ is a non-orthonormal basis state with Hermitian overlap matrix

$$K^2_{ij} = (i|j) \tag{4.10}$$

then

$$|j\rangle = K^{-1}|j) = |k)K^{-1}_{kj}, \tag{4.11}$$

where K is the Hermitian square root of K^2, is a member of the associated orthonormal basis. Furthermore, if

$$T|k) = |l)T_{lk}, \tag{4.12}$$

then

$$\begin{aligned} \langle i|T|j\rangle &= (i|K^{-1}TK^{-1}|j) \\ &= K_{il}T_{lk}K^{-1}_{kj}. \end{aligned} \tag{4.13}$$

In using (4.6), (4.7), (4.A – C) and (4.13) to obtain the reduced matrix elements of T, two alternatives arise:

(i) For cases (A1)-(B1),

$$[T^{\{10\}[10]}|\langle L\varepsilon\rangle\rangle]^{\{L'\varepsilon'\}[L'\varepsilon']} = F([L'\varepsilon'],[L\varepsilon])|\langle L'\varepsilon'\rangle\rangle. \tag{4.14}$$

We then obtain, for $L'+\varepsilon'=L+\varepsilon+1$,

$$\langle\{h'\}(n')[L'\varepsilon']\| T^{\{10\}[10]}\|\{h\}(n)[L\varepsilon]\rangle$$
$$=\sum_{n''} K_{n'n''}(\{h'\}[L'\varepsilon'])K^{-1}_{n''n}(\{h\}[L\varepsilon])\,U\left(\frac{1}{2}\frac{\sigma}{2}\frac{h'}{2}\frac{n''}{2};\frac{\sigma'}{2}\frac{h}{2}\right)F([L'\varepsilon'],[L\varepsilon]) \tag{4.15}$$

with the selection rule

$$n_1'+n_2'=n_1+n_2.$$

(ii) For cases (B2)-(C4),

$$[T^{\{10\}[10]}|\langle L\varepsilon\rangle\rangle]^{\{k\}[L'\varepsilon']}=F(\{k\};[L'\varepsilon'],[L\varepsilon])[Z^{\{20\}}(\mathscr{A})|\langle L'\varepsilon'\rangle\rangle]^{\{k\}}. \tag{4.16}$$

It follows that

$$[Z^{\{n\}}(\mathscr{A})[T^{\{10\}[10]}|\langle L\varepsilon\rangle\rangle]^{\{k\}[L'\varepsilon']}]^{\{h'\}}$$
$$=\sum_{n'}\mathrm{U}\left(\frac{n}{2}1\frac{h'}{2}\frac{\sigma'}{2};\frac{n'}{2}\frac{k}{2}\right)F(\{k\};[L'\varepsilon'],[L\varepsilon])[[Z^{\{n\}}(\mathscr{A})Z^{\{20\}}(\mathscr{A})]^{\{n'\}}|\langle L'\varepsilon'\rangle\rangle]^{\{h'\}}. \tag{4.17}$$

Now the tensors $Z^{\{n\}}$ are normalised such that

$$(0|Z^{\{n'\}}(a)Z^{\{n\}}(a^\dagger)|0)=\delta_{nn'} \tag{4.18}$$

where $a^\dagger$ is a boson creation operator of U(2) rank {20} and $|0)$ is the boson vacuum state. We conclude that

$$[Z^{\{n\}}(\mathscr{A})Z^{\{20\}}(\mathscr{A})]^{\{n'\}}=(0|Z^{\{n'\}}(a)[Z^{\{n\}}(a^\dagger)Z^{\{20\}}(a^\dagger)]^{\{n'\}}|0)Z^{\{n'\}}(\mathscr{A})$$
$$=(n'\|a^\dagger\|n)Z^{\{n'\}}(\mathscr{A}) \tag{4.19}$$

where $(n'\|a^\dagger\|n)$ is the U(2)-boson reduced matrix element given in the appendix of part I. We then obtain, for $L'+\varepsilon'=L+\varepsilon-1$,

$$\langle\{h'\}(n')[L'\varepsilon']\| T^{\{10\}[10]}\|\{h\}(n)[L\varepsilon]\rangle$$
$$=\sum_{mm'k}(-1)^{(\sigma'-k)/2}K_{n'm'}(\{h'\}[L'\varepsilon'])K^{-1}_{mn}(\{h\}[L\varepsilon])$$
$$\times\mathrm{U}\left(\frac{1}{2}\frac{\sigma}{2}\frac{h'}{2}\frac{m}{2};\frac{k}{2}\frac{h}{2}\right)\mathrm{U}\left(\frac{m}{2}1\frac{h'}{2}\frac{\sigma'}{2};\frac{m'}{2}\frac{k}{2}\right)(m'\|a^\dagger\|m)F(\{k\};[L'\varepsilon'],[L\varepsilon]). \tag{4.20}$$

5. Reduced matrix elements of the generators of SU(3) ⊃ SO(3)

The reduced matrix elements of the generators of SU(3) ⊃ SO(3) are defined by

$$\langle\{h'\}(n')[L'\varepsilon'];\alpha'M'|C_{lm}|\{h\}(n)[L\varepsilon];\alpha M\rangle$$
$$=\delta_{h'h}\delta_{\alpha'\alpha}\langle LM;lm|L'M'\rangle\langle\{h\}(n')[L'\varepsilon']\|C_l\|\{h\}(n)[L\varepsilon]\rangle. \tag{5.1}$$

The reduced matrix elements of L are trivially found to be given by

$$\langle\{h\}(n')[L'\varepsilon']\|L\|\{h\}(n)[L\varepsilon]\rangle=\delta_{L'L}\delta_{\varepsilon'\varepsilon}\delta_{n'n}[L(L+1)]^{1/2}. \tag{5.2}$$

For the quadrupole operator, we find

$$\langle\{h\}(n')[L'\varepsilon'];\alpha'M'|C_{2\nu}|\{h\}(n)[L\varepsilon];\alpha M\rangle$$
$$=\delta_{\alpha\alpha'}\sum_{\substack{h''n''L''M''\varepsilon''\alpha''\\ \beta mm'}}\sqrt{2}\langle 1m;1m'|2\nu\rangle$$
$$\times\langle\{h'\}(n')[L'\varepsilon'];\alpha'M'|T^{\{10\}[10]}_{\beta\ \ m}|\{h''\}(n'')[L''\varepsilon''];\alpha''M''\rangle$$
$$\times\langle\{h\}(n)[L\varepsilon];\alpha M|(-1)^{m'}T^{\{10\}[10]}_{\beta\ -m'}|\{h''\}(n'')[L''\varepsilon''];\alpha''M''\rangle, \qquad (5.3)$$

from which we easily deduce

$$\langle\{h\}(n')[L'\varepsilon']\|C_2\|\{h\}(n)[L\varepsilon]\rangle=\sum_{h''n''L''\varepsilon''}(-1)^{L+L''}\left(\frac{2(2L+1)}{(2L''+1)}\right)^{1/2}\mathrm{U}(11LL';2L'')$$
$$\times\langle\{h\}(n')[L'\varepsilon']\|T^{\{10\}[10]}\|\{h''\}(n'')[L''\varepsilon'']\rangle$$
$$\times\langle\{h\}(n)[L\varepsilon]\|T^{\{10\}[10]}\|\{h''\}(n'')[L''\varepsilon'']\rangle. \qquad (5.4)$$

Since

$$(C_{2\nu})^{\dagger}=(-1)^{\nu}C_{2-\nu}, \qquad (5.5)$$

we have

$$\langle\{h\}(n')[L'\varepsilon']\|C_2\|\{h\}(n)[L\varepsilon]\rangle$$
$$=(-1)^{L-L'}\left(\frac{(2L+1)}{(2L'+1)}\right)^{1/2}\langle\{h\}(n)[L\varepsilon]\|C_2\|\{h\}(n')[L'\varepsilon']\rangle. \qquad (5.6)$$

6. Examples

(i) $|\{h_1 0\}[L];\alpha M\rangle$

Since no degeneracy in L occurs in SU(3) symmetrical unirreps $\{h_1 0\}$ and since h_1-L is always even in such unirreps, we omit the multiplicity label and we use the simpler notation

$$|\{h_1 0\}[L];\alpha M\rangle \qquad (6.1)$$

to designate these states.

From (I5.4), we have

$$K^2(\{h_1 0\};[L])=(2L+3)(2L+5)\dots(h_1+L+1). \qquad (6.2)$$

Then from § 4, we easily obtain

$$\langle\{h_1+1,0\}[L+1]\|T^{\{10\}[10]}\|\{h_1 0\}[L]\rangle=\left(\frac{(h_1+L+3)(L+1)}{(2L+3)}\right)^{1/2} \qquad (6.3)$$

and

$$\langle\{h_1+1,0\}[L-1]\|T^{\{10\}[10]}\|\{h_1 0\}[L]\rangle=-\left(\frac{L(h_1-L+2)}{(2L-1)}\right)^{1/2}, \qquad (6.4)$$

(cf Wybourne 1974).

From (6.3), (6.4) and using (5.4), we then deduce

$$\langle\{h_1 0\}[L+2]\|C_2\|\{h_1 0\}[L]\rangle=\left(\frac{2(L+1)(L+2)(h_1-L)(h_1+L+3)}{(2L+3)(2L+5)}\right)^{1/2} \tag{6.5}$$

and

$$\langle\{h_1 0\}[L]\|C_2\|\{h_1 0\}[L]\rangle=-(2h_1+3)\left(\frac{L(L+1)}{3(2L-1)(2L+3)}\right)^{1/2}, \tag{6.6}$$

(cf Wybourne 1974).

(ii) $|\{42\}(n)[L\varepsilon];\alpha M\rangle$

The SU(3) unirrep {42} is the simplest example of a unirrep with an SO(3) multiplicity. The branching rule is given by

$$\mathrm{SU}(3)\downarrow\mathrm{SO}(3):\{42\}\downarrow[0]+2[2]+[3]+[4].$$

This unirrep has been thoroughly studied from the point of view of multiplicity in § 5.2 of part I to which we refer the reader.

A computer code has been written from which we retrieved the values given in table 1. These reduced matrix elements must satisfy some sum rules (Partensky and Quesne 1979). One of them is related to the quadratic Casimir operator of SU(3) given by

$$I_2=\sum_\alpha(-1)^\alpha L_\alpha L_{-\alpha}+\sum_\nu(-1)^\nu C_{2\nu}C_{2-\nu} \tag{6.7}$$

which takes the value

$$\langle I_2\rangle=\tfrac{4}{3}(h_1^2-h_1h_2+h_2^2+3h_1) \tag{6.8}$$

Table 1. Reduced matrix elements of C_2 in the {42} SU(3) unirrep.

$\langle\{42\}(42)[00]\|C_2\|\{42\}(42)[00]\rangle=0.0000$
$\langle\{42\}(40)[20]\|C_2\|\{42\}(42)[00]\rangle=1.1567$
$\langle\{42\}(22)[20]\|C_2\|\{42\}(42)[00]\rangle=2.2499$
$\langle\{42\}(40)[20]\|C_2\|\{42\}(40)[20]\rangle=2.5717$
$\langle\{42\}(40)[20]\|C_2\|\{42\}(22)[20]\rangle=1.8402$
$\langle\{42\}(22)[20]\|C_2\|\{42\}(22)[20]\rangle=-2.5717$
$\langle\{42\}(20)[31]\|C_2\|\{42\}(40)[20]\rangle=-3.0066$
$\langle\{42\}(20)[31]\|C_2\|\{42\}(22)[20]\rangle=1.5457$
$\langle\{42\}(20)[31]\|C_2\|\{42\}(20)[31]\rangle=0.0000$
$\langle\{42\}(20)[40]\|C_2\|\{42\}(40)[20]\rangle=1.0559$
$\langle\{42\}(20)[40]\|C_2\|\{42\}(22)[20]\rangle=2.0539$
$\langle\{42\}(20)[40]\|C_2\|\{42\}(20)[31]\rangle=-2.5820$
$\langle\{42\}(20)[40]\|C_2\|\{42\}(20)[40]\rangle=0.0000$

on a given $\{h_1h_2\}$ SU(3) unirrep. These equations imply

$$\sum_{n''L''\varepsilon''} \langle\{h\}(n)[L\varepsilon]\|C_2\|\{h\}(n'')[L''\varepsilon'']\rangle\langle\{h\}(n')[L\varepsilon]\|C_2\|\{h\}(n'')[L''\varepsilon'']\rangle$$
$$= \delta_{nn'}[\langle I_2\rangle - L(L+1)]. \qquad (6.9)$$

It can be verified that the values for the reduced matrix elements of C_3 given in table 1 satisfy this sum rule.

Another sum rule pertains to the trace of C_2 in a given L subspace of a $\{h_1h_2\}$ SU(3) unirrep. For a self-contragredient unirrep like {42}, for which

$$\{h_1, h_1 - h_2\} = \{h_1, h_2\},$$

the trace sum rule gives

$$\sum_n \langle\{h_1h_2\}(n)[L\varepsilon]\|C_2\|\{h_1h_2\}(n)[L\varepsilon]\rangle = 0, \qquad L \geqslant h_2.$$

Once again it can be verified that the values given in table 1 satisfy this sum rule. The sum rule for the trace in the general case can be found in Partensky and Quesne (1979).

References

Arima A and Iachello F 1976 *Ann. Phys., NY* **99** 253
Biedenharn L C, Giovannimi A and Louck J D 1967 *J. Math. Phys.* **8** 691
Chacón E, Castaños O and Frank A 1984 *J. Math. Phys.* **25** 1442
Deenen J and Quesne C 1983 *J. Phys. A: Math. Gen.* **16** 2095
Elliott J P 1958a *Proc. R. Soc.* A **245** 128
—— 1958b *Proc. R. Soc.* A **245** 562
Klimyk A U 1983 *J. Phys. A: Math. Gen.* **16** 3693
Le Blanc R and Rowe D J 1985 *J. Phys. A: Math. Gen.* **18** submitted for publication
Moshinsky M and Quesne C 1970 *J. Math. Phys.* **11** 1631
—— 1971 *J. Math. Phys.* **12** 1772
Partensky A and Quesne C 1979 *J. Math. Phys.* **20** 2014
Quesne C 1984a *J. Phys. A: Math. Gen.* **17** 777
—— 1984b *J. Phys. A: Math. Gen.* **17** 791
Rosensteel G and Rowe D J 1980 *Ann. Phys., NY* **126** 343
Wybourne B G 1974 *Classical Groups for Physicists* (New York: Wiley) ch 20